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Asymptotic quantum behavior of classically anomalous maps

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In the framework of quantum chaos, the theory of dynamical localization plays an outstanding role, both for its conceptual relevance and physical import. Theoretical arguments, confirmed by a large amount of numerical simulations, have shown in the case of complete classical chaos, that the localization length is related to the *classical* diffusion constant and the effective Planck's constant $\hbar$. We investigate the quantum behavior when classical dynamics exhibits anomalous diffusion (so that the diffusion constant is not defined): we show that dynamical localization still takes place, and that the scaling with the quantum parameter is the same as the classically diffusive case.

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One of the most remarkable phenomena in quantum chaos is represented by dynamical localization, namely the suppression of deterministic chaotic diffusion at large enough time scales [1]. The original formulation [2] considered the behavior of the kicked rotator, that is the quantum analog of the classical standard map [3]

$$\begin{aligned} p_{n+1} &= p_n - K\sin(x_n), \\ x_{n+1} &= x_n + p_{n+1}, \end{aligned} \tag{1}$$

which represents a paradigmatic example of the complexity of classical Hamiltonian dynamics, and catches the essential features of a number of nonlinear physical problems. One of the most striking classical properties of Eq. (1), for a wide range of large K values, is the appearance of *deterministic diffusion*, namely, if we consider Eq. (1) on a cylinder (the position x being actually an angle), we get

$$\sigma_2(n) = \langle (p_n - p_0)^2 \rangle \sim 2Dn, \tag{2}$$

where the average is over a set of initial conditions. The classical dynamics is thus described by a stochastic parameter—the diffusion constant D—(which depends on the nonlinear parameter K [4]). When the system is quantized the most striking consequence is that classical diffusion in p is suppressed (for large enough times), and the asymptotic distributions are exponentially localized in the momentum representation (see [1,2,5,6]). This phenomenon (called *quantum dynamical localization*) bears remarkable analogies to Anderson localization [7]: the essential physical parameter associated to the quantum motion is thus the *localization length* ξ, which gives the rate of exponential localization of the asymptotic wave function.

A remarkable theoretical argument [8] links, in the regime of small effective $\hbar$ and genuine classical deterministic diffusion (2), the localization length to both classical and quantum parameters

$$\xi = \alpha \frac{D}{\hbar^2}, \tag{3}$$

where extensive investigations [9] strongly indicate $\alpha = 1/2$. These considerations are not of purely theoretical significance, since quantum dynamical localization has been observed in a number of different experimental setups [10].

The aim of this paper is to analyze the asymptotic quantum regime when the above-mentioned scenario changes in an essential way at the classical level, that is when classical diffusion is *anomalous*. As a matter of fact, a number of recent studies [11] have pointed out the existence of a set of K values where classical transport does not follow the linear relationship (2), but rather

$$\sigma_2(n) \sim n^{\mu} \tag{4}$$

with some nontrivial exponent μ (see Fig. 1). Actually, not only the second moment of momentum distribution is anomalous, but the whole spectrum of moments is characterized by a scaling function $\beta(q)$, defined by the asymptotic relationship

$$\langle (p_n - p_0)^q \rangle = \sigma_q(n) \sim n^{q\beta(q)}. \tag{5}$$

Figure 2 shows how for $K = 6.905$ (which is the parameter value we focus on in our analysis) the scaling function $\beta(q)$ has a nontrivial profile, in contrast to genuine normal diffusing cases (the dashed line in Fig. 2).

Our choice of the nonlinearity parameter K is dictated by a pioneering analysis of the quantum motion [12], where, over the considered time scale, the authors observed a slowdown of the quantum growth law for $\sigma_2(n)$, but not the saturation that is a hallmark of dynamical localization. Our first task is thus an analysis, over long time scales, of the quantum dynamics corresponding to $K = 6.905$, with different values of the effective Planck's constant $\hbar$. We recall that the quantum dynamics involves the one-kick unitary evolution operator

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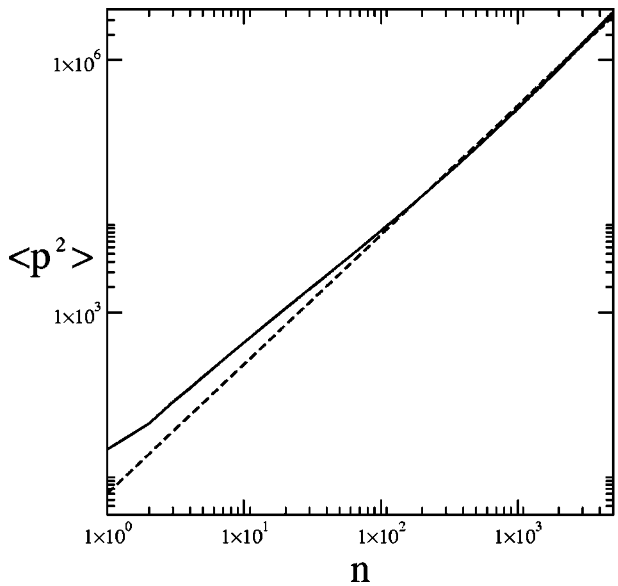

FIG. 1. $\langle p^2\rangle$ for the classical standard map ($K=6.905$): the average is over 10^7 initial conditions. The dashed line has a slope $\mu=1.5344$.

$$\hat{U}=e^{-(i/2\hbar)\hat{p}^2}e^{-(i/\hbar)K\cos(\hat{x})},\tag{6}$$

whose evolution is most conveniently studied by selecting a (sufficiently large) momentum basis (plane waves) and shifting back and forth (via a fast Fourier transform) to the conjugate representation so that the two (noncommuting) exponentials in Eq. (6) act as multiplications on the wave function [13]. Typically we start with an initial state localized around the zero-momentum state, and follow the evolution of differ-

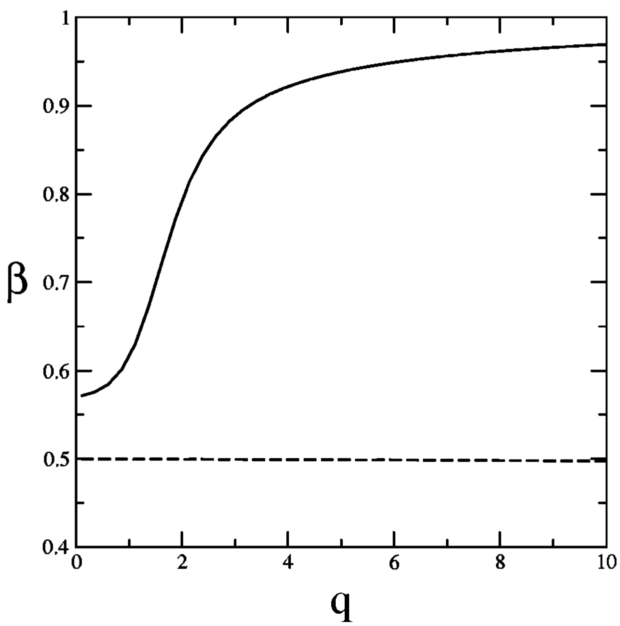

FIG. 2. The scaling indices $\beta(q)$ for $K=6.905$ (full line). Each exponent has been estimated by taking 2×10^5 initial conditions and 10^4 iterations of the map. The dashed line exhibits the same scaling indices for a normal diffusion case ($K=11$).

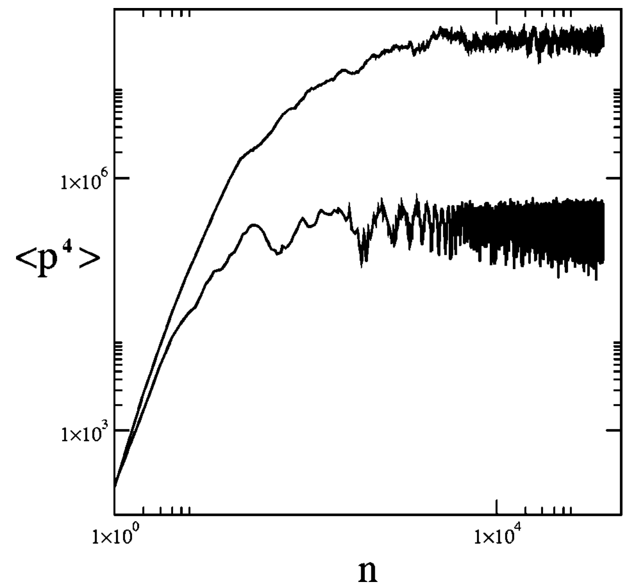

FIG. 3. Quantum evolution of $\langle p^4\rangle$ for $K=6.905$: the asymptotic localization is clearly exhibited. The upper curve refers to $\hbar=0.27$ and the lower one to $\hbar=2.7$.

ent order moments. As an example we show in Fig. 3 the evolution of the fourth moment for two different values of $\hbar$: the main numerical result is that asymptotically we *always* observe saturation, thus, even in presence of accelerated diffusion in the classical dynamics, the analysis of quantum spreading of the wave packet strongly indicates that asymptotically dynamical localization prevails. We remark that the choice of the basis size strongly depends on $\hbar$: for the values of $\hbar$ we considered (in the same regime as [12], where the quantization scale is larger than the phase-space volume occupied by ordered structures), basis sizes up to 3^{11} momentum states were used and we always checked that normalization of the wave function is preserved during the whole range of time evolution (ruled by the number of kicks n). We point out that our simulations refer to ''generic'' (nonresonant) values of $\hbar$: the influence of weak chaos on resonant [14] behavior has been recently considered by Alonso [15].

The remarks about asymptotic saturation of diffusion and exponentially localized distributions do not imply that for short and intermediate times we do not observe any deviation from the standard picture: in particular the ''ballistic peaks'' follow the behavior predicted in former analysis [16], and an interesting intermediate regime is indeed present [12], but here we address only the asymptotic behavior, and this is always found to be consistent with dynamical localization. The analysis of asymptotic distribution over momentum states further supports this conclusion, and a *pure exponential* decay is exhibited, see Fig. 4.

Once we convince ourselves that, despite the anomalous character of classical dynamics, the long-time quantum behavior is still ruled by exponential localization, a natural question involves scaling properties of the localization length, as the standard argument (3) loses its meaning because D is no more defined. We recall that in the standard theory, the way in which classical diffusion constant enters quantum dynamical localization is through an estimate of

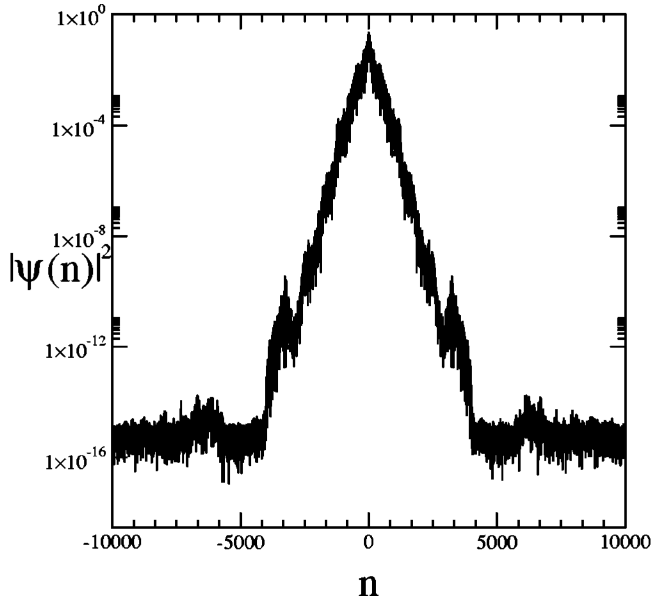

FIG. 4. Final distribution probability over momentum states for $K=6.905$ and $\hbar=0.599$.

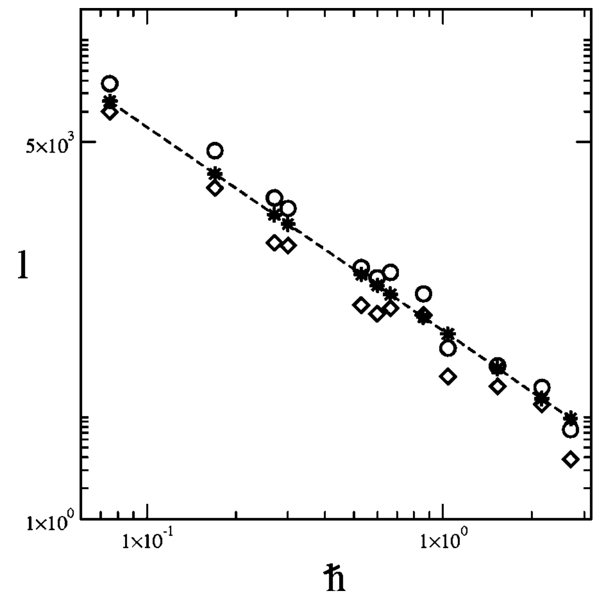

FIG. 5. Scaling of localization length versus $\hbar$, for $K=6.905$. Circles represent estimates from the entropy, stars are related to IPR, and diamonds from the slope of asymptotic distribution over momentum states. The dashed line has slope 2.

quantum states visited before quantum effects dominate: $\Delta n \sim (Dt)^{1/2}$. The anomalous character of classical diffusion then enters the argument directly, and in principle one might expect a completely new scaling behavior (with respect to $\hbar$, that links *n* to *p* momentum states). As a matter of fact, if one estimates the spreading by the square of the second moment one gets $\Delta n \sim t^{\mu/2}$. There is a still subtler issue, namely that the nontriviality of the classical moments' distribution (see Fig. 2) might lead to nonequivalent estimates of the localization length [if, for instance, we put $\Delta n_{(q)} = (\Delta |n|^q)^{1/q}$, these are ruled by different exponents] as from Eq. (5) we get $\Delta n_{(q)} \sim t^{\beta(q)}$, where sweeping *q* space we pick up an interval of possible exponents. So we checked a number of measures of ξ: first of all from a direct inspection on the asymptotic distribution, looking at an exponential decay rate $e^{-n/\xi}$. We then considered the inverse participation ratio [17]

$$IPR = \frac{\sum_n |a_n|^4}{\left(\sum_n |a_n|^2\right)^2}, \tag{7}$$

(a_n being the momentum bases coefficients of the wave function). We recall that the inverse participation ratio measures the inverse of the number of momentum states concurring to the distribution (one then gets $\xi_{IPR} \sim 1/2IPR$). We finally looked at the entropy [18]

$$S = \sum_n |a_n|^2 \log|a_n|^2. \tag{8}$$

Here ξ is thought to be proportional to the number of participating states $\xi_S \sim \exp(S)$. Our findings consistently suggest that every possible way of quantifying the localization length yields, within numerical errors, the same scaling with $\hbar$, as seen from Fig. 5.

Our simulations strongly indicate that, even in the presence of anomalies in the classical regime, dynamical localization still rules the asymptotic dynamics, and that the scaling with the quantum parameter $\hbar$, somehow surprisingly, still maintains the form of the normal diffusion case. The absence of corrections with respect to the standard picture is coherent with the observation that the most dramatic classical effects (ballistic peaks propagation) are exponentially suppressed in the quantum regime, but further work is needed to get a deeper understanding of the asymptotic regime.

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